Magnetosphere-Ionosphere Coupling by Alfvén waves at mid-latitudes

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Abstract

Numerical modeling of magnetosphere-ionosphere coupling by Alfvén waves has proven to be a valuable tool in describing the propagation and evolution of field-aligned currents and the magnetic fields produced by these waves as observed on the ground. Although many models of this type have assumed that magnetic field lines penetrate the ionosphere vertically, this assumption is not valid at lower latitudes where the dipole tilt is significant. This paper presents a new model of this interaction that takes the dipole geometry into account while including an ionosphere that is radially stratified. Since the orthogonal dipole coordinates that have been used in previous studies do not come to a constant radial distance at the ionosphere, a non-orthogonal system is developed that reduces to the orthogonal system at high altitudes. This model is useful for many different problems of wave coupling, and is applied here to the propagation of ducted Pc1 oscillations that can propagate 1000’s of kilometers across magnetic field lines.

Introduction

The evolution of field-aligned currents can be described in terms of the passage of shear-mode Alfvén waves along auroral field lines. The auroral flux tube is terminated by the conducting ionosphere, which reflects Alfvén waves unless the ionospheric Pedersen conductance is matched to the Alfvén wave impedance (e.g., Scholer, 1970; Goertz and Boswell, 1979). Moreover, a realistic auroral flux tube contains strong gradients in the Alfvén speed, with scale lengths less than typical Alfvén wavelengths. Such gradients cause partial reflection of the wave (Mallinckrodt and Carlson, 1978), which can lead to a frequency dependence of the reflection coefficients (Lysak, 1991).

The gradients in the Alfvén speed are particularly strong in the region up to about 1 RE altitude, since the mass density decreases exponentially with increasing altitude while the magnetic field falls off less rapidly. Thus, the Alfvén speed increases rapidly, reaching a peak at an altitude of about 1 RE that can be comparable to the speed of light. This sharp rise forms a resonant cavity, termed the ionospheric Alfvén resonator by Polyakov and Rapaport (1981) and studied extensively by Trakhtengertz and Feldstein (1984, 1991) and Lysak (1986, 1988, 1991, 1993). This cavity has resonant frequencies in the range of 0.1-1.0 Hz.

It has long been known from ground observations that Pc1 waves in this band are common in the ionosphere (e.g., Jacobs and Watanabe, 1962; Manchester, 1968; Fraser, 1975; Hansen et al., 1992; Popecki et al., 1993; Neudegg et al., 1995). Simultaneous observations of Pc1 waves by ground magnetometers and the Viking satellite have been reported by Arnoldy et al. (1988, 1996) and Potemra et al. (1992). Less structured waves in this band are the PiB bursts observed in the midnight sector associated with substorms (e.g., Heacock, 1967; Böninger et al., 1981; Koskinen et al., 1993). These PiB bursts are typically associated with the arrival of the
substorm current wedge at the ionosphere, and may be associated with conductivity enhancements due to localized precipitation (Grant and Burns, 1995).

Waves in this frequency range have frequently been observed by satellites and sounding rockets. Temerin et al. (1981) noted that S3-3 satellite observations of large quasi-static electric fields are consistent with being electrostatic structures at high altitudes; at altitudes below 5000 km; however, the observed electric fields were more consistent with being large amplitude Alfvén waves. Dynamics Explorer observations (Gurnett et al., 1984) have also indicated that low frequency electric and magnetic field observations are consistent with an Alfvén wave interpretation. Similar results have been obtained by the ICB-1300 satellite (Chmyrev et al., 1985), Aureol-3 (Berthelier et al., 1989), Magsat (Iyemori and Hayashi, 1989), HILAT (Knudsen et al., 1990, 1992), and sounding rockets (Boehm et al., 1990). Viking observations have shown that the peak power of electric and magnetic fluctuations occurs in this same frequency range (Marklund et al., 1990; Block and Fälthammar, 1990; Erlandson et al., 1990).

It is interesting to note that Volwerk et al. (1996) report a small compressional component in association with the Alfvén wave, consistent with a coupling between these two wave modes. Arnoldy et al. (1998) observed Pi1 waves simultaneously at the GOES spacecraft and on the ground. These observations indicated that although the general frequency response at the satellites and the ground were similar, there were significant differences both in the spectral width of the emissions and in the timing of the event, indicating that the wave signature was modified by wave propagation through the ionosphere.

New evidence on the propagation of Alfvén waves in the auroral zone has been given by FAST and Polar, as well as on sounding rockets. Sigsee et al. (1998) studied examples of FAST wave fields using wavelet analysis and observed waves near 0.9 Hz. They also observed lower frequency waves that were interpreted as a field-line resonance. Chaston et al. (1999, 2000, 2002a,b) observed a number of Alfvén wave events on FAST with ratios of electric to magnetic field that indicated that the perpendicular wavelength of the waves was comparable to the electron inertial length, and associated these waves with auroral particle acceleration. Ivchenko et al. (1999) have observed waves at about 0.6 Hz on the Auroral Turbulence II rocket, consistent with waves trapped in the ionospheric Alfvén resonator. These waves produced modulations in the precipitating electron flux (Lynch et al., 1999). Recently, observations of Pc1 pulsations at Polar have been correlated with ground observations (Mursula et al., 2001).

While the density structure above the ionosphere leads to a resonance cavity for shear Alfvén waves, it provides a waveguide for compressional waves that can propagate across field lines (e.g., Greifinger and Greifinger, 1968). Therefore, a signal observed on the ground may not be on the same field line as the field-aligned current structure that produced it. Neudegg et al. (1995) have shown from an array of ground magnetometers that Pc1 signals propagate with a typical speed of 450 km/s over distances of a few hundred kilometers, consistent with propagation in this ionospheric waveguide. Yahima et al. (2000) have noted that Pc1 oscillations can be seen a few hours of magnetic local time away from their source. These observations indicate that these waves can propagate over large distances through the ionosphere. Over such long distances, the magnetic field dip angle will change and so models that assume a vertical field will have to be modified. The early work of Greifinger and Greifinger (1968, 1973) indicates that waves polarized in the poloidal or toroidal directions will interact with the ionosphere differently due to dipole tilt. Fujita and Tamao (1988) studied the propagation of Pc1 waves in a simplified model, and also found changes in the polarization as
the wave propagated. These studies assumed a constant frequency and perpendicular wave number; our studies will extend this work to consider waves propagating in space and time.

This paper will present the first results from a model of MHD wave propagation above the ionosphere that includes the full effects of dipole tilt. This model is based on previous models of wave propagation (e.g., Lysak, 1999; Lysak and Song, 2001) that utilized a simplified dipole coordinate system in which the flux tube was assumed to maintain its shape, so that scale factors in the two perpendicular directions were the same. While this model is useful near the poles (indeed, it is valid for $r << L$, where $L$ is radial distance of the dipole field line where it crosses the equator), it fails to take into account the true dipolar geometry important at lower latitudes. Models of field line resonances in dipole coordinates (e.g., Radoski, 1967; Lee and Lysak, 1989, 1991; Rankin et al., 1993, 1994; Streltsov and Lotko, 1995, 1999) have been developed, but these models generally have ionospheres at artificially high altitudes. This is in part a consequence of the fact that the field-aligned coordinate of the dipolar coordinate system is not at a constant radial distance in general, while the ionosphere, being largely controlled by gravity, is radially stratified. Since it is desirable to have the ionospheric boundary at a constant radial distance while the magnetospheric dynamics is dominated by the magnetic field geometry, a non-orthogonal coordinate system that can take both geometries into account is useful. This paper will present first results from a model of this type, applied primarily to waves in the Pc1 band that are strongly affected by the structure of the ionosphere and the ionospheric waveguide/resonator immediately above it.

**Numerical Model**

The model used is based on the linearized MHD equations, which can be conveniently written in terms of Maxwell’s equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$  \hspace{1cm} (1)

$$\varepsilon_\perp \frac{\partial \mathbf{E}_\perp}{\partial t} = \frac{1}{\mu_0} (\nabla \times \mathbf{B})_\perp$$  \hspace{1cm} (2)

where $\varepsilon_\perp = \varepsilon_0 (1 + c^2 / V_A^2)$ is the low-frequency dielectric constant for a plasma. Note that in this model, parallel electric fields will be excluded since we are primarily dealing with large-scale fields for which the parallel electric field is negligible. Many previous models (e.g., Radoski, 1967; Lysak, 1985; Lee and Lysak, 1989, 1991; Rankin et al., 1993, 1994; Streltsov and Lotko, 1995, 1999) have cast these equations into orthogonal dipole coordinates, defined by

$$\nu = -\frac{R_E \sin^2 \theta}{r} \hspace{1cm} \varphi = \varphi \hspace{1cm} \mu = \frac{R_E \cos \theta}{r^2}$$  \hspace{1cm} (3)

Note that these coordinates are defined here to be dimensionless. The coordinates $\nu$ and $\varphi$ label a magnetic field line, while $\mu$ is the orthogonal coordinate that labels the position along the magnetic field line. It may be noted that the unit vectors along these coordinates are given by

$$\hat{\nu} = \hat{r} \sin \theta - \hat{\theta} 2 \cos \theta \hspace{1cm} \hat{\mu} = -\frac{\hat{r} 2 \cos \theta + \hat{\theta} \sin \theta}{\sqrt{1 + 3 \cos^2 \theta}}$$  \hspace{1cm} (4)

with the $\varphi$ unit vector being the same as in spherical or cylindrical coordinates. Equations (1) and (2) can be cast in this coordinate system using the scale factors.
Using these factors equations (1) and (2) can be written as

\[
\frac{\partial B_\nu}{\partial t} = -\frac{1}{h_\nu h_\mu} \frac{\partial}{\partial \mu} \left( h_\nu E_\phi \right)
\]

(6)

\[
\frac{\partial B_\phi}{\partial t} = \frac{1}{h_\nu h_\mu} \frac{\partial}{\partial \mu} \left( h_\nu E_\nu \right)
\]

(7)

\[
\frac{\partial B_\mu}{\partial t} = \frac{1}{h_\nu h_\phi} \left[ \frac{\partial}{\partial \nu} \left( h_\phi B_\phi \right) - \frac{\partial}{\partial \phi} \left( h_\nu B_\nu \right) \right]
\]

(8)

\[
\frac{\partial E_\nu}{\partial t} = \frac{V^2}{h_\nu h_\mu} \left[ \frac{\partial}{\partial \phi} \left( h_\mu B_\mu \right) - \frac{\partial}{\partial \mu} \left( h_\nu B_\nu \right) \right]
\]

(9)

\[
\frac{\partial E_\phi}{\partial t} = \frac{V^2}{h_\nu h_\mu} \left[ \frac{\partial}{\partial \mu} \left( h_\nu B_\nu \right) - \frac{\partial}{\partial \phi} \left( h_\mu B_\mu \right) \right]
\]

(10)

where \( V^2 = 1/\epsilon_\perp \mu_0 \) is the square of the Alfvén speed (including the displacement current correction important when the Alfvén speed is close to the speed of light). It should be noted that in the electrostatic case, equations (6) and (7) indicate that \( E_\phi \) and \( B_\nu \) are constant along field lines; i.e., these factors take into account the dipole mapping often invoked to compare fields at different locations.

These dipole coordinates are very useful in describing the propagation of MHD waves in the magnetosphere. The field-aligned and perpendicular coordinates are separated, and in the case of azimuthal symmetry (no \( \phi \) dependence), it can be easily seen that \( E_\nu \) and \( B_\phi \) represent the toroidal (shear Alfvén) mode, while \( E_\phi \), \( B_\nu \) and \( B_\mu \) give the poloidal (compressional) mode. However, a major drawback is that the field-aligned coordinate \( \mu \) does not correspond to a constant radial distance, except at the pole. Since the ionosphere, which constitutes the lower boundary of the region of interest, is largely controlled by gravity, it would be expected to be at a constant radial distance. At lower latitudes, the magnetic field lines do not enter the ionosphere at right angles, as has been assumed in some earlier work (e.g., Lysak, 1997, 1999; Lysak and Song, 2001). This tilted field configuration must be included for a realistic model of magnetosphere-ionosphere coupling at mid-latitudes.

In order to include such effects, it is convenient to introduce a non-orthogonal coordinate system, as has recently been discussed by Proehl et al. (2002). Rather than introduce a numerical procedure as was done in that work, here we will simply modify the dipole coordinates to make the field-aligned coordinate surface be at a constant radial distance \( R_I \) at the ionospheric boundary. Procedures for writing the basic equations in non-orthogonal coordinates have been presented by d’Haeseleer et al., (1991), together with the basic notions of differential geometry. This process distinguishes between the covariant (written with a subscript) and contravariant (written with a superscript) components of a vector. The coordinates are written in terms of their contravariant components, as follows

\[
u^1 = -\frac{R_I}{r} \sin^2 \theta \quad \nu^2 = \phi \quad \nu^3 = \frac{R_I^2 \cos \theta}{r^2 \cos \theta_o}
\]
Note that the only change from the orthogonal dipole coordinates, other than the use of the ionospheric radial distance $R_I$ rather than the Earth’s radius $R_E$, is the introduction of the invariant co-latitude at the ionosphere $\theta_0$ in the field-aligned coordinate $u^3$. This invariant co-latitude can be written as $\cos \theta_0 = \sqrt{1 + u^3} = \sqrt{1 - R_I / R}$. It can be seen that $u^3 = \pm 1$ at the two ionospheres ($r = R_I, \theta = \theta_0$ or $\theta = \pi - \theta_0$), and $u^3 = 0$ at the equator ($\theta = \pi/2$).

As discussed by Proehl et al. (2002), the equations (1) and (2) can be cast into these coordinates using the contravariant and covariant basis vectors. The contravariant basis vectors are defined by $e^i = \nabla e_i$, and represent vectors that are normal to the planes defined by $u_i = \text{constant}$ (and so are sometimes called the normal basis vectors). The covariant basis vectors are defined by $e_i = \partial / \partial u^i$, where $\partial_i$ will be used to denote $\partial / \partial u^i$. These vectors are tangent to the $u^i$ coordinate curve, i.e., the curve defined by the other two coordinates being constant. Thus, these are referred to as the tangential basis vectors. Note that these are not unit vectors, but they are reciprocal to each other in the sense that $e^i \cdot e_j = \delta^i_j$. Applying these definitions to the coordinates given by (11), we find:

$$e^1 = \frac{R_I}{r^2} \sin \theta \left( \sin \theta \hat{r} - 2 \cos \theta \hat{\theta} \right)$$

$$e^2 = \frac{1}{r \sin \theta} \hat{\phi}$$

$$e^3 = -\frac{R_I^2}{r^3 \cos^3 \theta_0} \left[ \frac{\cos \theta}{2} \left( 1 + 3 \cos^2 \theta_0 \right) \hat{r} + \sin \theta \left( 1 - \frac{R_I}{r} \right) \hat{\theta} \right]$$

$$e_1 = \frac{r^2}{R_I \cos^2 \theta_0} \left[ \left( 1 - \frac{R_I}{r} \right) \hat{r} - \cot \theta \left( 1 + 3 \cos^2 \theta_0 \right) \hat{\theta} \right]$$

$$e_2 = r \sin \theta \hat{\phi}$$

$$e_3 = -\frac{R_I^2}{r^3 \cos \theta_0} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right]$$

Note that since the coordinates have been defined to be dimensionless, the contravariant basis vectors have a dimension of inverse length while the covariant basis vectors have a dimension of length. It can also be seen that the normal basis vectors $e^1$ and $e^2$ are perpendicular to the magnetic field, while the tangential basis vector $e_3$ is parallel to the magnetic field (Note that $e^1$ is parallel to $\hat{\nu}$ and $e_3$ is parallel to $\hat{\mu}$). At the ionosphere ($r = R_I$), it can be seen that $e_1$ and $e_2$ are directed in latitude and longitude, respectively, while $e_3$ is radial. Finally, to relate these vectors to the orthogonal coordinates, it can be noted that the magnitude of the vectors $e^1$ and $e^2$ are given by $1/h_\nu$ and $1/h_\theta$, respectively, and the magnitude of $e_3$ is $h_\mu$ (which now contains an additional factor of $\cos \theta_0$ compared to its value in the orthogonal system).

The contravariant and covariant components of a vector $A$ can be written as $A^i = A \cdot e^i$ and $A_i = A \cdot e_i$, respectively. These components are related by the metric tensor $g_{ij} = e_i \cdot e_j$ by the relation $A_i = g_{ij} A^j$, where the sum over the repeated upper and lower indices is implied. Note that the only off-diagonal elements of the metric tensor in the present system are $g_{13}$ and $g_{31}$. The contravariant metric tensor can be defined in an analogous way. Now the basic equations
(1) and (2) can be written with the help of the standard definition of a curl (d’Haeseleer et al., 1991):

$$\nabla \times \mathbf{A} = \frac{1}{J} \varepsilon^{ijk} e^i \partial_j A_k$$  \hspace{1cm} (18)

where $\varepsilon^{ijk}$ is the anti-symmetric permutation tensor (0 if any two indices are the same, ±1 for cyclic/anticyclic permutations of the indices). Here $J = e_1 \cdot (e_2 \times e_3) = \sqrt{\det g}$ is the Jacobian for the system, which gives the volume element $dV = J du^1 du^2 du^3$. Explicit forms of these functions are given in the Appendix. These quantities allow us to write the basic equations (6)-(10) in the form

$$\frac{\partial B^1}{\partial t} = \frac{1}{J} \partial_3 E_2 \quad \frac{\partial B^2}{\partial t} = -\frac{1}{J} \partial_3 E_1 \quad \frac{\partial B^3}{\partial t} = \frac{1}{J} (\partial_2 E_1 - \partial_1 E_2)$$  \hspace{1cm} (19)

$$\frac{\partial E^1}{\partial t} = \frac{V^2}{J} (\partial_3 B_3 - \partial_1 B_2) \quad \frac{\partial E^2}{\partial t} = \frac{V^2}{J} (\partial_3 B_1 - \partial_1 B_3)$$  \hspace{1cm} (20)

Note that since this is ideal MHD, the parallel electric field component $E_3$ is zero. Thus, after using equations (19) and (20) to evolve the contravariant components, the new covariant components can be determined by

$$E_1 = E^1 / g^{11} \quad E_2 = g^{22} E^2$$

$$B_1 = g_{11} B^1 + g_{13} B^3 \quad B_2 = g^{22} B^2 \quad B_3 = g_{31} B^1 + g_{33} B^3$$  \hspace{1cm} (21)

so that there is no need to follow the final component $E^3$. These equations are solved by a leapfrog method on a staggered grid, which is set up so that all of the differences required are centered. Periodic boundary conditions are assumed in the longitudinal ($u^2$) direction. In the latitudinal ($u^1$) direction, the boundary conditions assume that $B_1$ and $E_2$ are zero, as is the derivative $\partial_1 B_3$. Note that the other fields are internal to this boundary and do not require explicit boundary conditions. At the top of the simulation (the $u^3$ boundary), an input pulse is assumed, and the effective conductivity is set to $1/\mu_0 V_d$. This condition allows waves incident on this top boundary to pass out of the system (e.g., Lysak, 1985).

For these first results, a slab model for the ionosphere is adopted. This slab is assumed to be at the constant radial distance $R_i$. Integrating Ampere’s Law across this layer leads to a relation that has been previously used by a number of authors (e.g., Fujita and Tamao, 1988; Yoshikawa and Itonaga, 1996, 2000; Lysak and Song, 2001; Sciffer and Waters, 2002)

$$\mu_0 \hat{\mathbf{r}} \times \mathbf{E} = \hat{\mathbf{r}} \times \Delta \mathbf{B}$$  \hspace{1cm} (22)

where $\Delta \mathbf{B}$ is the difference in the magnetic field components above and below the layer and $\hat{\mathbf{r}}$ is the height-integrated conductivity tensor. Note that this equation relates the horizontal components of the magnetic field and height-integrated current, rather than the components perpendicular to the background magnetic field. Assuming that the vertical current is zero in this model (see Kelley, 1989; Sciffer and Waters, 2002), the conductivity tensor can be written in the form

$$\Sigma = \begin{pmatrix}
\frac{\Sigma_0 \Sigma_p}{\Sigma_{zz}} & -\frac{\Sigma_0 \Sigma_H \cos \alpha}{\Sigma_{zz}} \\
\frac{\Sigma_0 \Sigma_H \cos \alpha}{\Sigma_{zz}} & \frac{\Sigma_p + \Sigma_H^2 \sin^2 \alpha}{\Sigma_{zz}}
\end{pmatrix}$$  \hspace{1cm} (23)
where $\Sigma_P$, $\Sigma_H$, and $\Sigma_0$ are the height-integrated Pedersen, Hall, and parallel conductivities, respectively, $\alpha$ is the angle between the magnetic field and the vertical, given by $\cos \alpha = -2 \cos \theta / \sqrt{1 + 3 \cos^2 \theta}$, and $\Sigma_{zz} = \Sigma_0 \cos^2 \alpha + \Sigma_P \sin^2 \alpha$. It should be noted that while this model is a reasonable representation of the thin E-layer of the ionosphere, the F-layer is more extended in altitude and cannot be represented by a thin shell at Pc1 frequencies. Strictly speaking, the F-layer should be modeled by including collisional terms in equation (2) as was done in Lysak (1997, 1999); however, for simplicity those collision terms are set to zero in the runs presented in this paper.

The ionospheric boundary condition is implemented as follows. The ionospheric boundary condition (22) is written in terms of the spherical components of the vector, and so involves the components $E_1$, $E_2$, $B_1$, $B_2$, and $B_3$. In the staggered grid, the electric field components and the $B^3$ component of the magnetic field are defined at the ionospheric boundary value of $u^i$, while the 1 and 2 components of the magnetic field are a half cell above. The magnetic field components can all then be calculated by the usual equations (19). Thus, in order to invert equation (22) in order to find the electric field, the magnetic field components below the ionosphere must be determined. These are found by assuming that the atmosphere below the ionosphere is a perfect insulator, i.e., no currents flow in the atmosphere, and that the ground is a perfect conductor. Then since $\nabla \times B = 0$ and $\nabla \cdot B = 0$, the magnetic field can be written in terms of a scalar potential, $B = \nabla \Psi$, where $\nabla^2 \Psi = 0$. The radial component of the magnetic field is continuous across the ionospheric layer, and so the radial boundary conditions on $\Psi$ are $\partial \Psi / \partial r = B_r$ at the ionosphere (where $B_r = h_r B^3$, as in equation (A32)) and $\partial \Psi / \partial r = 0$ at the ground, where the radial magnetic field perturbation is zero due to the perfectly conducting ground. The details of the solution to Laplace’s equation are given in the Appendix.

Results

We wish to apply the model described above to the propagation of Pc1 oscillations above the ionosphere. We model the region from $L = 2$ to $L = 20$ (invariant latitudes of 45° to 77°) and consider 180° of longitude. The model extends from the ionosphere (at 200 km altitude, or $R_I = 1.0314 R_E$) to a point which is at a radial distance of 4 $R_E$ on the $L = 10$ field line (Note that this boundary is a surface of constant $u^3$, and so is at 1.96 $R_E$ for $L = 2$ and 4.27 $R_E$ for $L = 20$). A meridional cut through the simulation volume is shown in Figure 1. In this figure, the dashed lines give some representative curves of $u^1$ (essentially the L-shells) and $u^3$, the field-aligned coordinate. Note that in subsequent figures, this surface is mapped onto a rectangle, and labeled in terms of the invariant latitude and the radial distance along the $L = 10$ field line. Note also that the runs shown were performed on a grid with 128 cells in the two perpendicular directions and 75 cells in the parallel direction.

The background magnetic field is the usual dipole field of the Earth, and the density profile is given by a commonly used form of an exponential (assumed to be oxygen) plus a power law (assumed to be hydrogen):

$$n(r) = n_o e^{-(r-R_i)/h} + n_H r^{-p}$$

Note that this gives the particle density; for the mass density (used to calculate the Alfvén speed) the first term is multiplied by the oxygen mass and the second by the hydrogen mass. We consider two types of profiles: in the first, we take the parameters to be $n_o = 10^5$ cm$^{-3}$, $n_H = 20$
cm$^{-3}$, $h = 400$ km (or 0.0627 $R_E$) and $p = 1$. This model approximates parameters in the auroral zone and polar cap (e.g., the density is 10 cm$^{-3}$ at 6000 km altitude), so we will refer to this as the “auroral” profile (although note that the density in this model is larger than in the so-called auroral density cavity where the density can be <1 cm$^{-3}$ at 6000 km). The second density profile used includes the plasmaspheric plasma, in which all parameters are the same as the auroral profile except the hydrogen density, which is increased to $n_H = 1000$ cm$^{-3}$. These density profiles and the corresponding Alfvén speed profiles are shown in Figure 2. In the runs shown below, the auroral profile is used for latitudes greater than 60° ($L = 4$), while the plasmasphere profile is used at lower latitudes. Note that the transition between the two regions is modeled as a hyperbolic tangent with a scale length $\Delta L = 0.5$. Since the cell spacing in this region is $\delta L \sim 0.05$, this transition is well resolved.

The runs are initialized with a 1 Hz perturbation introduced at the upper boundary of the system. This perturbation takes the form of a Gaussian profile in the electrostatic potential; therefore, the driven wave is purely in the shear mode, with the electric field radially into or out of the center of the perturbation. This produces a wave in which the magnetic perturbation circles the center of the perturbation, leading to a co-axial current structure in which a field-aligned current at the core of the perturbation is balanced by a return current around the sides. For the runs shown, the center of the perturbation is at $L = 8$ and a longitude of 90° (Note that the zero point of longitude is arbitrary since the model has cylindrical symmetry). With the density profile used, this pulse will take about 2 s to reach the ionosphere. We will consider a case where the Pedersen conductance is 1 mho and the Hall conductance is 2 mho for most of the results shown; we will contrast the results with a higher conductivity case below.

Figures 3 and 4 show the shear wave propagation for a high conductivity case where the auroral profile is assumed for all field lines. Figure 3 shows the 4 perpendicular field components at a radial distance of 1.64 $R_E$ (i.e., near the FAST apogee of 4000 km). The perspective in this figure is that the observer is looking upward from the ground. These fields are given in physical units in field-aligned coordinates; thus, in these plots $B_v = B^1 h_v$, $B_\phi = B^2 h_\phi$, and $B_\mu = B^3 h_\mu$ and similarly for the electric field. Note that in the following, altitudes or radial distances are set at the $L = 10$ field line; thus, these planes give the fields at the value of the coordinate $u^3$ that corresponds to 2 $R_E$ at $L = 10$. The horizontal and vertical axes give the invariant latitude and longitude for each field line. Lighter shaded regions (red and green in color versions of the plots) give positive values of the variable while darker regions (blue in color versions) give negative values. Thus, this figure shows electric fields directed radially outward from the center of the pulse while the magnetic fields circulate counter-clockwise around the center, giving a downward field-aligned current. The polarity of these fields reverses a half second later for this 1 Hz wave. Figure 4 shows cuts through the center of the pulses at constant longitude for $E_v$ and $B_\phi$ and at constant latitude for $E_\phi$ and $B_v$. These plots indicate that a 3$^{rd}$ harmonic of the Alfvén resonator has been excited. Note that we excite a higher harmonic of the resonator since the fundamental does not couple to the waveguide modes (Lysak and Song, 2001). The electric field is largest near the Alfvén speed maximum, while the magnetic field is larger near the ionosphere where the Alfvén speed is small.

Next we shall focus on the fields in the Alfvén resonator/waveguide region. Figure 5 shows the four perpendicular field components at a time of 6 seconds into the run and a radial distance of 1.16 $R_E$, corresponding to 1000 km altitude, a typical altitude for sounding rocket observations. In each of these plots, the maximum contour level has been set so that the waves radiating in the waveguide are brought out. The maximum value of each field and the maximum
contour level is given for each plot. To aid in the interpretation of these plots, Figure 6 shows contours of constant distance from the central field line at the ionosphere. It can be seen that the wave fronts of the fields roughly follow these contours, and that the wavelength of the wave is the order of 1000 km. Note also that there are small perturbations in the $E_{\nu}$ and $B_{\phi}$ components at 60° latitude. These correspond to the excitation of a weak surface wave at the plasmapause. Figure 7 shows three cuts of the compressional magnetic field component $B_{\mu}$, with the top panel showing the fields at 1000 km, while the middle and bottom panels are cuts at a latitude of 69° and a longitude of 100°, respectively. These plots illustrate that a compressional magnetosonic wave has been excited and propagates away from the central field line ($L = 8$ or $\Lambda = 69.3^\circ$, $\phi = 90^\circ$). Figure 8 shows a series of snapshots of the compressional magnetic field every 0.25 s from 3.25 s to 4.5 s. This figure shows the propagation of the compressional wave away from the source field line. Again comparing with Figure 6, it can be seen that the wave fronts propagate at just over 1000 km/s, consistent with the 1000 km wavelength and 1 s period of this wave.

To see a more quantitative version of this simulation, the electric and magnetic fields at this altitude and time are plotted as a function of longitude in Figure 9. These cuts are at an invariant latitude of 67°, just off the central line. In these figures, the solid curves represent the $\nu$ component, the dotted curves the $\phi$ component, and (for the magnetic field plots) the dashed curves give the $\mu$ component. Note that the large central peak in the electric field, which goes up to 160 mV/m in $E_{\nu}$ and 90 mV/m in $E_{\phi}$, has been cut off. It can be seen that the waves have an amplitude of about 5 mV/m in the electric fields and 5 nT in the magnetic field, giving a ratio the order of 1000 km/s. Note from Figure 2 that this corresponds to the Alfvén velocity in the region near the ionosphere. It may be further noted that the fields in the main shear Alfvén pulse exhibit a larger ratio, i.e., $E_{\nu} \approx 160$ mV/m and $B_{\phi} \approx 10$ nT, giving a ratio of 16,000 km/s. This higher ratio illustrates the non-local nature of the resonator structure; since the shear wave fields extend to higher altitudes the ratio is indicative of a higher value of the Alfvén speed.

Next we consider the ground perturbations produced by this wave. As noted above, the ground magnetic fields can be determined by taking the horizontal gradient of the magnetic scalar potential given by equation (A34). Note that the vertical component of the magnetic field is set to zero by our boundary conditions (perfect conductivity of the ground). Figure 10 shows the $\theta$ and $\phi$ components of the ground magnetic field at $t = 6$ s, the same time as the fields shown in Figures 5 and 6. The dominant ground magnetic field is directly under the source field line, but it can be seen that evidence of the weak field corresponding to the propagation in the ionospheric waveguide can be seen. Here the peak fields on the source field line are the order of 30 nT, with the propagating fields are up to 5 nT. Figure 11 shows the same plot for a case that is identical to the run shown above but with conductivities of $\Sigma_P = 5$ mho and $\Sigma_H = 10$ mho. Now the maximum fields in the source region are 4 nT, while the ducted wave is the order of 0.1 nT. This conductivity dependence is likely due to the fact that the ground signature is associated with the vertical magnetic field component caused by the curl of the horizontal electric fields. Thus, at high conductivity when these electric fields are weak, the ground magnetic signature is correspondingly weak. This implies that ducted waves should be easier to observe on the ground for nighttime conditions rather than in daylight.

One somewhat surprising feature that can be seen in Figure 10 and 11 is that the wave fields appear to propagate more strongly to the east (higher longitude) than to the west. This effect also appears upon careful examination of the fields in the waveguide itself (e.g., Figures 7, 8 and 9). This effect was not seen in our previous model that considered only vertical field lines (Lysak and Song, 2001), and so appears to be an interaction of the Hall effect with the
inhomogeneity associated with the dipole tilt. To see this, consider how the coupling between the shear and compressional modes occurs in the ionosphere. For the incident shear wave in the co-axial configuration shown here, the electric field points radially inward, which drives a radial Pedersen current, which in turn gives an azimuthal Hall electric field. The curl of this electric field drives a vertical magnetic field perturbation: note that it is $B^3$, which is related directly to $B_r$ in the ionosphere by equation (A32) that appears in the curl equation (19). The structure of this vertical magnetic component (plotted in Figure 12) reflects the symmetry of the current structure. However, since the parallel electric field is zero, the longitudinal Poynting flux depends on the parallel magnetic field perturbation, which is directly related by $B_3$ by equation (A31); i.e., $S_\phi = -E_v B_\mu / \mu_0$. This parallel magnetic field is a superposition of the vertical magnetic field and the latitudinal component of the magnetic field, $B_3 = g_{31} B^1 + g_{33} B^3$, or in terms of the physical field components, $B_r = B_\mu \cos \alpha + B_\nu \sin \alpha$. As can be seen from Figure 12, this rotation introduces an asymmetry into the $B_\mu$ component so that $E_v$ and $B_\mu$ become anticorrelated, leading to a net positive $S_\phi$. Thus, this combination of the Hall effect and dipole tilt leads to the eastward propagation. While this calculation is done assuming northern hemisphere parameters, this effect has the same sign in the southern hemisphere since both the Hall effect and the tilt angle change signs due to the change in magnetic field polarity.

**Conclusions**

The results presented here give an indication of the usefulness of a model that describes the propagation of MHD waves through the magnetosphere, including dipole tilt effects that have been missing in previous models. This model, based on a non-orthogonal dipole coordinate system, improves upon previous models by including a more realistic ionospheric boundary that is radially stratified while keeping the dipole geometry in the magnetosphere. While the mathematical development of the model is rather involved, the implementation is relatively straightforward since the governing equations (19) and (20) have the same structure as the orthogonal dipole equations (6)-(10). Indeed, the only additional work that must be done each time step in the bulk of the simulation is the rotation between contravariant and covariant components given by equations (21). The other major addition to the code is the boundary condition, which is now based on a spherical harmonic expansion rather than the Fourier expansion used in previous models; however, this does not entail a significant amount of additional computational time per time step.

The results from this model indicate that Pc1 pulsations can indeed propagate at speeds the order of 1000 km/s across magnetic field lines, covering a large range around the Earth in only a few seconds. This implies that the ionospheric waveguide can be an effective conduit for transferring energy globally. This confirms results obtained with simpler numerical models of the Pc1 waveguide (e.g., Lysak and Song, 2001). One unanticipated result is that the propagating fast mode waves give a preference for eastward rather than westward propagation due to the interaction between the Hall currents and the dipole tilt effects. The study of Neudegg et al. (1995) showed observations indicating that Pc1 waves were observed propagating from the region of the cusp. Of the 42 events in this study, it appears from their Figure 2 that 22 came from the west (indicating eastward propagation) while 20 came from the east. Thus, this dataset
does not show a significant east/west asymmetry, although most of the events observed at higher Kp indeed came from the west. Further study may shed more light on whether such propagation asymmetries can be observed in ground-based data.

One aspect of this model that deserves further consideration is the effect of waves propagating into different regions of the magnetosphere. In this model, we have included a plasmaspheric model at low latitudes. This does not appear to have much effect on the propagation of waves in the Pc1 waveguide, since this cavity is largely unaffected by the higher density at higher altitude in the plasmasphere as opposed to higher latitudes. Runs (not shown) that have changed the parameters of the lower ionospheric density model do show a change in the wave phase velocity that is consistent with the change of the Alfvén speed in this region. Further work will consider models in which the ionospheric parameters can also change, for example, at the terminator between the sunlit and dark regions of the model.

The model presented here has concentrated on mid-latitudes, since the coordinate system used has singularities at the pole and at the equator. A more complete model can be constructed by including modules for the polar and equatorial regions. Near the pole, the dipole tilt effects become negligible, and a simplified dipole coordinate system such as that used in Lysak and Song (2001) would be applicable. Near the equator, the dipole field lines shrink to zero length, and so a field-aligned coordinate system is more difficult to implement. In this region, we can abandon the concept of a field-aligned system and model the equations in normal spherical coordinates. These improvements on the present model will be left for future work.
Appendix: Mathematical details of the model.

It can be seen from the basic definitions given in the text that the non-zero metric tensor elements can be written as

\[
g_{11} = \frac{R_f^2}{r^4} \sin^2 \theta \left(1 + 3 \cos^2 \theta \right)
\]

\[
g_{13} = g_{31} = \frac{R_f^3}{r^5} \sin^2 \theta_0 \cos \theta \left(1 + 3 \cos^2 \theta \right)
\]

\[
g_{22} = \frac{1}{r^2 \sin^2 \theta}
\]

\[
g_{33} = \frac{R_f^4}{r^6 \cos^6 \theta} \left[ \frac{\cos^2 \theta \left(1 + 3 \cos^2 \theta_0 \right)^2}{4} + \sin^2 \theta \left(1 - \frac{R_f}{r} \right)^2 \right]
\]

and the covariant components are

\[
g_{11} = \frac{r^4}{R_f^2 \cos^4 \theta_0 \left(1 + 3 \cos^2 \theta \right)^2} \left[ \left(1 - \frac{R_f}{r} \right)^2 + \frac{\cot^2 \theta \left(1 + 3 \cos^2 \theta_0 \right)^2}{4} \right]
\]

\[
g_{13} = g_{31} = \frac{r^4}{2R_f^2 \cos \theta_0 \left(1 + 3 \cos^2 \theta \right)}
\]

\[
g_{22} = r^2 \sin^2 \theta
\]

\[
g_{33} = \frac{r^6 \cos^2 \theta_0}{R_f^4 \left(1 + 3 \cos^2 \theta \right)}
\]

The Jacobian is given by

\[
J = \frac{r^6 \cos \theta_0}{R_f^4 \left(1 + 3 \cos^2 \theta \right)}
\]

It is worth noting that the modified dipole scale factors as in equation (5) are given by \( h_v = 1/\sqrt{g_{11}} \), \( h_\theta = \sqrt{g_{22}} = 1/\sqrt{g_{33}} \), and \( h_\mu = \sqrt{g_{33}} \), and the Jacobian can be written as \( J = h_v h_\theta h_\mu \). With these definitions, it can be seen that the dipole unit vectors are related to the basis vectors by

\[
\hat{v} = h_v e^1 \quad \hat{\phi} = h_\theta e^2 = e_2 / h_\theta \quad \hat{\mu} = e_3 / h_\mu
\]

It is also important to note that at the ionosphere, the other basis vectors simplify, in particular

\[
e_1 = -\frac{R_f}{2 \sin \theta_0 \cos \theta_0} \hat{\theta} \quad e^3 = -\frac{1 + 3 \cos^2 \theta_0}{2R_f \cos^2 \theta_0} \hat{r}
\]

Thus, at the ionosphere, the spherical scale factors can be introduced

\[
h_0 = -\frac{R_f}{2 \sin \theta_0 \cos \theta_0} \quad h_r = -\frac{2R_f \cos^2 \theta_0}{1 + 3 \cos^2 \theta_0}
\]

These scale factors are significant since they allow for the contravariant and covariant components of the electric and magnetic field vectors to be converted into “physical” field...
components with the proper units and scaling. Thus we can write the perpendicular and parallel components of the perturbed magnetic field as

\[ B_\perp = \mathbf{B} \cdot \hat{\mathbf{v}} = h_B B^1 \quad B_\parallel = h_B B^2 = B_2 / h_\parallel \quad B_\mu = B_3 / h_\mu \]  

(A31)

and the same for the electric field. Similarly, at the ionosphere, the spherical components of the magnetic field are

\[ B_0 = B_1 / h_0 \quad B_\theta = h_r B^3 \]  

(A32)

As noted in the text, the magnetic fields in the atmosphere can be computed from a magnetic scalar potential \( \Psi \) that satisfies Laplace’s equation. Since the radial component of the magnetic field is continuous across the ionospheric layer, the radial boundary conditions on \( \Psi \) are \( \partial \Psi / \partial r = B_r \) at \( r = R_I \) (where \( B_r = h_B B^3 \), as in equation (A32)) and \( \partial \Psi / \partial r = 0 \) at \( r = R_E \), where the radial magnetic field perturbation is zero due to the perfectly conducting ground. The solution to Laplace’s equation in the atmosphere can then be solved with a spherical harmonic expansion. However, since the model does not cover the whole sphere, a full spherical harmonic expansion would require too many terms. The lower latitude boundary has the boundary condition \( B_\theta = 0 \), and so \( \partial \Psi / \partial \theta = 0 \) at this boundary. This implies that only Legendre functions that satisfy this boundary condition are needed. Thus, the eigenfunctions for Laplace’s equation in this domain depend on Legendre functions of degree \( \nu \) that satisfy the boundary condition \( \partial P_\nu^m (\cos \theta_{\text{low}}) / \partial \theta = 0 \) (see, e.g., Jackson, 1999, section 3.4). So, denoting the \( l^{th} \) solution to this equation as \( y_l \), the eigenfunctions can be written as

\[ y_{lm}(\theta, \phi) = C_{lm} P_\nu^m (\cos \theta) e^{im\phi} \]  

(A33)

where the functions are denoted by a lower case \( y \) to distinguish them from the usual spherical harmonics, and where \( C_{lm} \) are normalization constants, which are determined numerically. Thus, the general solution to Laplace’s equation can be written as

\[ \Psi(r, \theta, \phi) = \sum_{l,m} \left( A_{lm} r^\nu_l + B_{lm} r^{-(\nu_l+1)} \right) y_{lm}(\theta, \phi) \]  

(A34)

We may further note that at \( r = R_E \), the boundary condition on \( \Psi \) requires that

\[ B_{lm} = \frac{\nu_l}{\nu_l + 1} R_E^{2\nu_l + 1} A_{lm} \]  

(A35)

The boundary condition at \( r = R_I \), the ionospheric radial distance, then determines the coefficients \( A_{lm} \)

\[ A_{lm} = \frac{1}{\nu_l R_I^{\nu_l - 1} \left[ 1 - \left( R_E / R_I \right)^{2\nu_l + 1} \right]} \int d\Omega \ B_r (R_I, \theta, \phi) y_{lm}^* (\theta, \phi) \]  

(A36)

(Note that there is no problem with the possibility that \( \nu_l = 0 \) since this can only occur for \( m = 0 \), since \( \nu_l \geq m \), and so we have \( y_{00} = \text{constant} \), which gives no contribution to the fields once the gradient is taken.) Once these coefficients are obtained, the horizontal magnetic field at the ionosphere can be determined for use in equation (22), and this equation can then be inverted to find the horizontal electric field. In addition, the magnetic perturbation at the ground can be evaluated by taking the gradient of (A34) evaluated at \( r = R_E \).

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References:


Figure Captions

Figure 1. A cut through the simulation volume in a meridional plane. Note that the heavy lines indicate the boundaries of the simulation volume in the runs shown. Dashed lines give curves of constant $u^1$ and $u^3$.

Figure 2. Electron density and Alfvén speed profiles for the model used in the simulations. The solid line refers to the profile used at high latitudes (> 60°), while the dotted line is used at low latitudes to model the plasmasphere.

Figure 3. Profiles of the perpendicular field components $E_\nu$, $E_\phi$, $B_\nu$ and $B_\phi$ at a time 3 seconds after the start of the run and an altitude of about 4000 km, showing the basic form of the propagating Alfvén wave.

Figure 4. Profiles of the field components through the center of the wave pulse at the same time as in Figure 3. The four plots give $E_\nu$ at a longitude of 90°, $E_\phi$ at a latitude of 69°, $B_\nu$ at a latitude of 69°, and $B_\phi$ at a longitude of 90°. Note the third harmonic structure along the magnetic field line.

Figure 5. Plots of the perpendicular field components at a time 6 seconds from the start of the run at an altitude of 1000 km. Note the propagation of the signals away from the source region (at latitude 69° and longitude 90°). The maximum contour levels have been set at 10 mV/m for the electric fields and 5 nT for the magnetic fields to bring out the propagating wave.

Figure 6. Plot of contours of equal distance as measured in the ionosphere from the source at latitude of 69° (or $L = 8$) and longitude 90°. Note that the fields in Figure 5 correspond roughly to this figure.

Figure 7. Plots of the parallel $B_\mu$ component of the magnetic field at a time 6 seconds after the start of the run. Top panel gives contours at an altitude of 1000 km, middle panel gives the field component at 69° latitude and the bottom panel shows $B_\mu$ at 90° longitude. Note that in the bottom two panels the maximum contour level has been set at 10 nT to show the propagating fields.

Figure 8. Plots of the compressional $B_\mu$ component at an altitude of 1000 km every 0.25 seconds from 3.25 s to 4.5 s, showing the propagation of the fast mode wave in time.

Figure 9. Line plots of the electric (top) and magnetic (bottom) fields at an altitude of 1000 km and a latitude of 67° at 6 seconds. Note that the solid lines give the $\nu$ (latitudinal) component, the dotted lines give the $\phi$ (longitudinal) component and, for the magnetic field, the dashed line gives the $\mu$ (parallel) component.
Figure 10. Magnetic field perturbations as observed on the ground at 6 seconds into the run. The top panel gives the $B_\theta$ (southward) component while the bottom panel gives the $B_\phi$ (eastward) component. Note that the maximum contour level is set at 5 nT.

Figure 11. Ground magnetic fields for a similar run with the Pedersen and Hall conductivities set at $\Sigma_P = 5$ mho and $\Sigma_H = 10$ mho. Note now the maximum contour level is at 0.5 nT, showing that higher conductivity reduces the ground fields.

Figure 12. Field components important for the eastward propagation of the Pc1 ducted signal. Top left panel gives the vertical magnetic field $B_r$, top right panel gives the parallel magnetic field $B_\mu$, and the bottom panels give the latitudinal fields $E_\nu$ and $B_\nu$. The combination of $B_r$ and $B_\nu$ gives an asymmetric pattern to $B_\mu$, which combined with $E_\nu$ gives an eastward component to the Poynting flux.
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